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PROXIMATE ORDER AND TYPE OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

BY

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ABSTRACT

We introduce the notion of proximate order for entire functions with index-pair (p, q), for integers p and q such that $1 \leq q \leq p$ to study the type of entire functions of several complex variables. One of our main results is a generalization of a theorem due to Lelong–Gruman in their book [7] on Entire Functions of Several Complex Variables.

1. Introduction

The notions of order and type of entire functions are classical in complex analysis. For the entire functions of one complex variable, G. Valiron [11] refined these growth scales by introducing intermediate comparison functions, called proximate orders, which make it unnecessary to consider functions of minimal or maximal type. At that time, functions of more than two complex variables were mostly unknown to complex analysts (with the exception of H. Poincaré) and G. Valiron was not aware of the technical advantage given by such a notion which gives an

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indicatrix if the type is zero or infinity. For entire functions of several complex variables the advantage of this method stems from the fact that only functions of normal type will have non-trivial indicator functions which give the growth in different directions.

This paper deals with the following problem: Let $f(z); z \in \mathbb{C}^n$ be an entire transcendental function and further let $\Gamma(z)$ be a norm (see section 2 for definition) exhuasting the space $\mathbb{C}^n$ by a family of domains $\{D(r)\}_{r=0}^{\infty}$ where $D(r) = \{z \in \mathbb{C}^n : \Gamma(z) \leq r\}$; $r \in \mathbb{R}$. Find necessary and sufficient conditions for $f(z)$ to be of (p,q)-proximate order $\rho(r)$; $r \in \mathbb{R}$ with corresponding (p,q)-type σ in terms of a suitably defined sequence (introduced by A. A. Goldberg [2]) of coefficients associated with $f(z)$. The novelity in our approach consists of using the extension of the classical notions of (p,q)-orders and (p,q)-types to generalize results of Lelong–Gruman in the entire functions of several complex variables. Recall that the concepts of (p,q)-orders and (p,q)-types for entire functions of one complex variable were introduced by Juneja et al. [3,4]. The growth of an entire function can be studied in terms of its (p,q)-orders and (p,q)-types. However, these concepts are inadequate for comparing the growth of those entire functions which are of the same (p,q)-orders but of infinite (p,q)-types. To refine the above scale the notion of proximate order was developed by Nandan et al. [10] for entire functions of one complex variable with index-pair (p,q).

In this note these ideas on (p,q)-proximate order are extended to entire functions of several complex variables instead of the usual $(1,1)$ one. The modifications needed in calculations are immediate even in this general case. In particular, we prove Theorem 1.18 of [7] in a general setting without the use of Mean-Value Theorem and extend their Theorem 1.23 to gap power series of homogeneous polynomials on the (p,q)-scale. Applications of the main theorem will appear elsewhere. The explicit formulae obtained in this note are of independent interest as to the best of our knowledge we have not seen them in any relevent literature prior to this work.

2. Preliminaries and auxiliary results

Let $\Gamma\colon \mathbb{C}^n \to \mathbb{R}^+ = [0,\infty[$ be a real-valued function such that the following conditions hold:

(1) $\Gamma(z+w) \leq \Gamma(z) + \Gamma(w) \quad \forall z, w \in \mathbb{C}^n$

(2) $\Gamma(\lambda z) = |\lambda|\Gamma(z) \quad \forall \lambda \in \mathbb{C}$

(3) $\Gamma(z) = 0 \Leftrightarrow z = 0$, then Γ is a norm.

Given a function; $\vartheta\colon \mathbb{C}^n \to \mathbb{R}^+$, consider the maximum of ϑ;

$M_{\vartheta,\Gamma}(r) = \mathrm{Sup}_{\Gamma(z)\leq r}\vartheta(z), \quad \forall\, r \in \mathbb{R}^+$ with respect to the norm $\Gamma(z)$.

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be an entire function, we say that f is of order ρ if $\log|f|$ is of order ρ, where

$$\rho = \limsup_{r\to\infty} \frac{\log M_{f,\Gamma}(r)}{\log r}.$$

If $\rho < +\infty$, $f(z)$ is said to be of maximal, normal or minimal type according as the quantity

$$\sigma = \limsup_{r\to\infty} \frac{M_{f,\Gamma}(r)}{r^\rho}$$

is $+\infty$, finite or zero and σ is said to be the type of f with respect to the norm $\Gamma(z)$.

For comparing the growth of those entire functions $f(z)$ on $\mathbb{C}^n$, which have their type as infinity, the concept of proximate order is used.

Definition 2.1: A proximate order $\rho(r)$ is a function defined for $r \in \mathbb{R}^+$ such that: (i) $\lim_{r\to\infty}\rho(r) = \rho$ and (ii) $\lim_{r\to\infty}\rho'(r)r\log(r) = 0$, where $\rho'(r)$ denotes the derivative of $\rho(r)$. Moreover, if

$$\sigma = \limsup_{r\to\infty} \frac{M_{f,\Gamma}(r)}{r^{\rho(r)}}$$

is finite and positive, then $\rho(r)$ is said to be the proximate order of the given function $f(z)$. ∎

Remark 2.2: Order and type for entire functions of two complex variables, in the classical setting, were introduced independently by Bose and Sharma [1] and Goldberg [2], who obtained their coefficient characterizations. However, the abstract approach to these ideas was developed by Lelong–Gruman [7]. Note that $\rho(r)$ is not unique. ∎

Let us introduce the notion of index-pair (p, q) for entire functions of several complex variables but first;

Definition 2.3: An entire function defined on $\mathbb{C}^n$ is said to be of (p,q)-**order** ρ if

$$\rho \equiv \rho(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M_{f,\Gamma}(r)}{\log^{[q]} r},$$

where $\log^{[m]} x = \exp^{[-m]}x = \log^{[m-1]} x) = \exp(\exp^{[-m-1]}x)$, $m = 0, \pm1, \pm2, ..$ provided that $0 < \log^{[m-1]} x < \infty$ with $\log^{[0]} x = \exp^{[0]}x = x$. ∎

Definition 2.4: An entire function $f(z)$ defined on $\mathbb{C}^n$ is of **index-pair** (p,q), $p \geq q \geq 1$ if $b < \rho(p,q) < \infty$ and $\rho(p-1,q-1)$ is not a finite nonzero number, where $b = 1$ if $p = q$ and $b = 0$ if $p > q$. If $\rho(p,p)$ is never greater than 1 and $\rho(p',p') = 1$ for some integer $p' \geq 1$, then the index-pair of $f(z)$ is defined as (m,m) where $m = \inf\{p' \colon \rho(p',p') = 1\}$. If $\rho(p,q)$ is never nonzero, finite and $\rho(p'',1) = 0$ for some integer $p'' \geq 1$, then the index-pair of $f(z)$ is defined as $(n,1)$ where $n = \inf\{p'' \colon \rho(p'',1) = 0\}$. If $f(z)$ is of index-pair (p,q) then $\rho \equiv \rho(p,q)$ is called its (p,q)-**order**. ∎

Definition 2.5: A positive function $\rho(r)$ defined on $[r_0, \infty[$, $r_0 \geq \exp^{[q-1]}1$, is a **proximate order** of an entire function on $\mathbb{C}^n$ with index-pair (p,q) if (i) $\rho(r) \to \rho(p,q) \equiv \rho$ as $r \to \infty, b < \rho < \infty$; (ii) $\Lambda_{[q]}(r)\rho'(r) \to 0$ as $r \to \infty$, $b = 1$ if $p = q$, $b = 0$ if $p > q$ and $\Lambda_{[q]}(r) = \prod_{i=0}^{q} \log^{[i]} r$. If in addition to (i) and (ii), we have (for $b < \rho < \infty$)

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_{f,\Gamma}(r)}{(\log^{[q-1]} r)^{\rho(r)}} = \sigma(p,q) \equiv \sigma, \quad 0 \leq \sigma \leq \infty,$$

then $\rho(r)$ is said to be the proximate order of the given entire function $f(z)$ if σ is nonzero and finite. ∎

The existence of such a comparison function has recently been established by Kasana [5] and Kasana and Sahai [6].

LEMMA 2.6: *$(\log^{[q-1]}r)^{\rho(r)-A}$ is a monotone increasing function of r for $0 < r_0 < r < \infty$, where $A = 1$ if $q = 2$ and zero otherwise.*

The easy proof of this Lemma which consists of showing that

$$\frac{d}{dr}((\log^{[q-1]} r)^{\rho(r)-A}) > 0, \quad \forall r > r_0,$$

can be found in [10]. Since $(\log^{[q-1]} r)^{\rho(r)-A}$ is a monotone increasing function of r, we can define a real function $\psi(x)$ of $x > x_0$ to be the unique solution of the equation

$$x = (\log^{[q-1]} r)^{\rho(r)-A} \Leftrightarrow \psi(x) = \log^{[q-1]} r. \tag{2.1}$$

LEMMA 2.7: *For the function $\psi(x)$ defined in (2.1) we have*

$$\lim_{x\to\infty} \frac{\psi(\eta x)}{\psi(x)} = \eta^{\frac{1}{\rho - A}} \quad \textit{uniformly for } 0 < \eta < \infty.$$

Proof: By the definition of proximate order and (2.1), we have

$$\frac{d[\log \psi(x)]}{d[\log x]} = \frac{1}{\rho(r) - A + \Lambda_{[q]}(r)\rho'(r)},$$

or

$$\lim_{x\to\infty} \frac{d[\log \psi(x)]}{d[\log x]} = \frac{1}{\rho - A}.$$

Now, for a given $\varepsilon > 0$, $x > x_0$ and $\eta > 1$ (the Lemma is trivial for $\eta = 1$),

$$(2.2) \quad \int_x^{\eta x} \left(\frac{1}{\rho - A} - \varepsilon\right) d[\log t] < \int_x^{\eta x} d[\log \psi(t)] < \int_x^{\eta x} \left(\frac{1}{\rho - A} + \varepsilon\right) d[\log t],$$

or

$$\left(\frac{1}{\rho - A} - \varepsilon\right) \log \eta < \log \frac{\psi(\eta x)}{\psi(x)} < \left(\frac{1}{\rho - A} + \varepsilon\right) \log \eta,$$

or

$$(2.3) \quad \eta^{\left(\frac{1}{\rho - A} - \varepsilon\right)} < \frac{\psi(\eta x)}{\psi(x)} < \eta^{\left(\frac{1}{\rho - A} + \varepsilon\right)}.$$

For $0 < \eta < 1$, (2.2) can be written as

$$\int_{\eta x}^{x} \left(\frac{1}{\rho - A} - \varepsilon\right) d[\log t] < \int_{\eta x}^{x} d[\log \psi(t)] < \int_{\eta x}^{x} \left(\frac{1}{\rho - A} + \varepsilon\right) d[\log t],$$

or

$$\left(-\frac{1}{\rho - A} + \varepsilon\right) \log \eta < \log \frac{\psi(x)}{\psi(\eta x)} < -\left(\frac{1}{\rho - A} + \varepsilon\right) \log \eta,$$

or

$$(2.4) \quad \eta^{-\left(\frac{1}{\rho - A} - \varepsilon\right)} < \frac{\psi(x)}{\psi(\eta x)} < \eta^{-\left(\frac{1}{\rho - A} + \varepsilon\right)}.$$

Taking limits in (2.3) and (2.4) and combining the outcomes the required result follows. ∎

Let $\{n_k\}$ be an increasing sequence of positive integers such that $n_k \to \infty$ as $k \to \infty$, and let $f(z)$ be an entire function on $\mathbb{C}^n$. We consider $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$, the Taylor series expansion of $f(z)$ in terms of homogeneous polynomials $P_{n_k}\colon \mathbb{C}^n \to \mathbb{C}$ of degree n_k. Define $C_{n_k} = \operatorname{Sup}_{\Gamma(z)\le 1}|P_{n_k}(z)|$ and the maximum term $\mu_{f,\Gamma}(r) = \sup_{k\ge 0}\{C_{n_k} r^{n_k}\}$, $r > 0$. Then, we have

LEMMA 2.8: *If $f(z)$ is an entire function on $\mathbb{C}^n$ of (p,q)-order $\rho (b < \rho < \infty)$ and (p,q)-type σ with respect to a proximate order $\rho(r)$, then*

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} \mu_{f,\Gamma}(r)}{(\log^{[q-1]} r)^{\rho(r)}} = \sigma,$$

where $\mu_{f,\Gamma}(r)$ is the maximum term of $f(z)$.

Proof: We recall a recent result of Lockhart–Straus [9], which can easily be extended to polydiscs such that

$$\mu_{f,\Gamma}(r) < M_{f,\Gamma}(r) < \left(\frac{4r+\varepsilon}{\varepsilon}\right)\left(1 + \frac{\log \mu_{f,\Gamma}(r+\varepsilon)}{\mu_{f,\Gamma}(r)}\right)\mu_{f,\Gamma}(r).$$

The Lemma is an immediate consequence of the above inequality. ∎

3. Main results

Definition 3.1: A function $L(r)$ defined for $r > 0$ is said to be **slowly increasing** if for every compact subinterval **I** of $]0,\infty[$ and for every $\varepsilon > 0$ there exists r_0 such that for $r > r_0$,

$$\left|\frac{L(kr)}{L(r)} - 1\right| < \varepsilon$$

for every $k \in \mathbf{I}$. ∎

THEOREM 3.2: *If $\rho(r)$ is a proximate order, then $(\log^{[q-1]} r)^{\rho(r)-\rho}$ is a slowly increasing function of r.*

Proof: Set

$$L(r) = (\log^{[q-1]} r)^{\rho(r)-\rho}.$$

Then

$$\log L(r) = (\rho(r) - \rho)\log^{[q]} r.$$

By differentiation we have

$$\frac{L'(r)}{L(r)} = \frac{\rho(r)-\rho}{\Lambda_{[q-1]}(r)} + \rho'(r)\log^{[q]} r. \tag{3.1}$$

Now from (3.1) we obtain

$$\Lambda_{[q-1]}(r)\frac{L'(r)}{L(r)} = \rho'(r)\Lambda_{[q]}(r) + (\rho(r) - \rho). \tag{3.2}$$

The right hand side of (3.2) tends to zero as $r \to \infty$, by the definition of proximate order. Hence

$$\lim_{r\to\infty} \Lambda_{[q-1]} \frac{L'(r)}{L(r)} \to 0.$$

Thus for given $\varepsilon > 0$ there exists r_0 such that for $r > r_0$ and for $1 < k < \infty$ we have

$$-\int_r^{kr} \frac{\varepsilon}{\Lambda_{[q-1]}(r)} < \int_r^{kr} \frac{L'(r)}{L(r)} < \int_r^{kr} \frac{\varepsilon}{\Lambda_{[q-1]}(r)}.$$

This implies that

$$-\varepsilon \left(\log^{[q]} kr - \log^{[q]} r\right) < \log \frac{L(kr)}{L(r)} < \varepsilon \left(\log^{[q]} kr - \log^{[q]} r\right),$$

and further,

$$\text{(3.3)} \qquad -\varepsilon \log \left(\frac{\log^{[q-1]} kr}{\log^{[q-1]} r}\right) < \log \frac{L(kr)}{L(r)} < \varepsilon \log \left(\frac{\log^{[q-1]} kr}{\log^{[q-1]} r}\right).$$

For all values of k such that $0 < k < 1$, we have

$$\text{(3.4)} \qquad -\varepsilon \log \left(\frac{\log^{[q-1]} r}{\log^{[q-1]} kr}\right) < \log \frac{L(r)}{L(kr)} < \varepsilon \log \left(\frac{\log^{[q-1]} r}{\log^{[q-1]} kr}\right).$$

Taking limits in (3.3) and (3.4) the assertion is thus clear for all k with $0 < k < \infty$ as

$$\log \frac{L(kr)}{L(r)} \to 0 \quad \text{when } r \to \infty$$

or

$$\lim_{r\to\infty} \frac{L(kr)}{L(r)} = 1,$$

which is just Definition 3.1. ∎

THEOREM 3.3: *Let $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$ be an entire function of several complex variables of (p,q)-order ρ ($b < \rho < \infty$) and (p,q)-type σ with respect to any proximate order $\rho(r)$. Then*

$$\limsup_{k\to\infty} \left[\frac{\psi(\log^{[p-2]} n_k)}{\log^{[q-1]} C_{n_k}{}^{-1/n_k}}\right]^{\rho - A} = \frac{\sigma}{M},$$

where

$$M = \begin{cases} (\rho-1)^{\rho-1}/\rho^{\rho}, & \text{if } (p,q) = (2,2), \\ 1/\rho e, & \text{if } (p,q) = (2,1), \\ 1, & \text{otherwise,} \end{cases}$$

and $P_{n_k}(z)$ *are homogeneous polynomials of degree* n_k.

Proof: By the definition of σ, we have for any given $\varepsilon > 0$, and $r > r_0$,

$$\log^{[p-1]} M_{f,\Gamma}(r) < (\sigma+\varepsilon)(\log^{[q-1]} r)^{\rho(r)}.$$

Set $\sigma + \varepsilon = \sigma'$, then

$$\log M_{f,\Gamma}(r) < \exp^{[p-2]}\sigma'(\log^{[q-1]} r)^{\rho(r)}.$$

By Cauchy's inequality $C_{n_k} r^{n_k} \le M_{f,\Gamma}(r)$ and hence

$$\log C_{n_k} < \exp^{[p-2]}(\sigma'(\log^{[q-1]} r)^{\rho(r)}) - n_k \log r. \tag{3.5}$$

Now choose r such that

$$(\log^{[q-1]} r)^{\rho(r)-A} = \frac{1}{\sigma'} \log^{[p-2]}(n_k/\rho). \tag{3.6}$$

For $(p,q) = (2,1)$, we follow Lelong–Gruman [7] to get

$$\limsup_{k\to\infty} \left[\psi(n_k) C_{n_k}^{1/n_k}\right]^{\rho} \le \sigma e \rho. \tag{3.7}$$

Consider $(p,q) = (2,2)$ and observe that (3.5) and (3.6) are reduced to

$$\log C_{n_k} < \sigma'(\log r)^{\rho(r)} - n_k \log r$$

and

$$(\log r)^{\rho(r)-1} = n_k/\sigma'\rho \Leftrightarrow \psi\left(\frac{n_k}{\sigma'\rho}\right) = \log r.$$

Hence

$$\begin{aligned}\log C_{n_k} &< \sigma'(\log r)^{\rho(r)} - n_k \log r \\ &= \log r\{\sigma'(\log r)^{\rho(r)-1} - n_k\} \\ &= \left(\frac{1}{\rho} - 1\right) n_k \log r,\end{aligned}$$

or

$$\log C_{n_k}^{-1/n_k} < \psi\left(\frac{n_k}{\rho\sigma'}\right)\left(\frac{1-\rho}{\rho}\right).$$

Thus multiplying both sides by $\psi(n_k)$ and simplifying we get

$$\frac{\psi(n_k)}{\log C_{n_k}^{-1/n_k}} < \frac{\rho\psi(n_k)}{(\rho-1)\psi(\frac{n_k}{\rho\sigma'})} \approx \frac{\rho}{\rho-1}(\rho\sigma')^{1/(\rho-1)}.$$

This yields

$$\left[\frac{\psi(n_k)}{\log C_{n_k}^{-1/n_k}}\right]^{\rho-1} < \left(\frac{\rho}{\rho-1}\right)^{\rho-1} \rho\sigma'.$$

Taking limits, we obtain

$$\limsup_{k\to\infty}\left[\frac{\psi(n_k)}{\log C_{n_k}^{-1/n_k}}\right]^{\rho-1} \le \frac{\rho^\rho\sigma'}{(\rho-1)^{\rho-1}} = \frac{\sigma}{M}. \tag{3.8}$$

For $(p,q)\neq(2,1)$ and $(p,q)\neq(2,2)$, observe that

$$(\log^{[q-1]} r)^{\rho(r)} = \frac{1}{\sigma'}\log^{[p-2]} n_k/\rho$$

if and only if

$$\psi\left(\frac{\log^{[p-2]} n_k/\rho}{\sigma'}\right) = \log^{[q-1]} r.$$

Now

$$\log C_{n_k} < \frac{n_k}{\rho} - n_k \exp^{[q-2]}\psi\left(\frac{\log^{[p-2]} n_k/\rho}{\sigma'}\right),$$

or

$$\log C_{n_k}^{-1/n_k} > (1+o(1))\exp^{[q-2]}\psi\left(\frac{\log^{[p-2]} n_k/\rho}{\sigma'}\right).$$

So we obtain

$$\log^{[q-1]} C_{n_k}^{-1/n_k} > \psi\left(\frac{\log^{[p-2]} n_k/\rho}{\sigma'}\right),$$

or

$$\frac{\psi(\log^{[p-2]} n_k)}{\log^{[q-1]} C_{n_k}^{-1/n_k}} < \frac{\psi(\log^{[p-2]} n_k/\rho)}{\psi(\frac{\log^{[p-2]} n_k/\rho}{\sigma'})}.$$

Thus

$$\limsup_{k\to\infty}\left[\frac{\psi(\log^{[p-2]} n_k)}{\log^{[q-1]} C_{n_k}^{-1/n_k}}\right]^{\rho} \le \sigma. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), it is easily seen that for all index-pairs (p,q)

$$\limsup_{k\to\infty}\left[\frac{\psi(\log^{[p-2]} n_k)}{\log^{[q-1]} C_{n_k}^{-1/n_k}}\right]^{\rho-A} \le \frac{\sigma}{M}. \tag{3.10}$$

To prove the reverse inequality, we let β to be a number such that $\beta < \sigma$ and

$$\limsup_{k\to\infty}\left[\frac{\psi(\log^{[p-2]} n_k)}{\log^{[q-1]} C_{n_k}^{-1/n_k}}\right]^{\rho-A} = \frac{\beta}{M}.$$

Thus, for given $\varepsilon > 0$ and $k > k_0$, we have

$$\psi(\log^{[p-2]} n_k) < \left(\frac{\beta+\varepsilon}{M}\right)^{1/(\rho-A)} \log^{[q-1]} C_{n_k}^{-1/n_k}.$$

Set $\beta + \varepsilon = \alpha$, then

$$\log^{[q-1]} C_{n_k}^{-1/n_k} > \left(\frac{M}{\alpha}\right)^{1/(\rho-A)} \psi(\log^{[p-2]} n_k),$$

or

$$C_{n_k} < \exp\left[-n_k \exp^{[q-2]}\left(\left(\frac{M}{\alpha}\right)^{1/(\rho-A)} \psi(\log^{[p-2]} n_k)\right)\right],$$

or

$$C_{n_k} r^{n_k} < \exp\left[-n_k \exp^{[q-2]}\left(\left(\frac{M}{\alpha}\right)^{\rho-A} \psi(\log^{[p-2]} n_k)\right) + n_k \log r\right].$$

So

$$\log C_{n_k} r^{n_k} < \sup_{k\geq 0}\left[-n_k \exp^{[q-2]}\left(\left(\frac{M}{\alpha}\right)^{1/(\rho-A)} \psi(\log^{[p-2]} n_k)\right) + n_k \log r\right].$$

Hence

$$\log \mu(r) < \sup_{k\geq 0}\left[n_k \log r - n_k \exp^{[q-2]}\psi\left(\frac{M}{\alpha}\log^{[p-2]} n_k\right)\right]. \tag{3.11}$$

Now for $(p,q) \neq (2,1)$ and $(p,q) \neq (2,2)$, the maximum is attained for

$$n_k = \exp^{[p-2]}\left[\alpha(\log^{[q-2]} r/e)^{\rho(r)}\right].$$

Using this value in (3.11), it follows that

$$\log \mu(r) \leq n_k.$$

Thus

$$\log^{[p-1]} \mu(r) < \alpha(\log^{[q-1]} r/e)^{\rho(r)}.$$

Now it is clear that

$$\frac{\log^{[p-1]}\mu(r)}{(\log^{[q-1]} r)^{\rho(r)}} < \alpha \left(\frac{\log^{[q-1]} r/e}{\log^{[q-1]} r} \right)^{\rho(r)} .$$

Taking limits we obtain (in view of Lemma 2.8)

$$\sigma \leq \alpha = \beta + \varepsilon. \tag{3.12}$$

So for $(p, q) = (2, 1)$, (3.11) is reduced to

$$\log \mu(r) \leq \sup_{k \geq 0} \left[\log r - \log \psi \left(\frac{n_k}{\rho e \alpha} \right) \right] n_k$$

and the maximum on the right hand side is attained for

$$n_k = \rho \alpha r^{\rho(r)},$$

so we have

$$\log \mu(r) \leq \frac{n_k}{\rho},$$

or

$$\frac{\log \mu(r)}{r^{\rho(r)}} \leq \alpha,$$

or

$$\sigma \leq \alpha. \tag{3.13}$$

Consider the case for $(p, q) = (2, 2)$ and the maximum in (3.11) in this case is attained for

$$n_k = \frac{\alpha}{M} (\frac{\rho - 1}{\rho})^{\rho(r)-1} (\log r)^{\rho(r)-1}.$$

Hence (3.11) is reduced to

$$\frac{\log \mu(r)}{(\log^{[q-1]} r)^{\rho(r)}} \leq \frac{\alpha}{M} \left(\frac{\rho - 1}{\rho} \right)^{\rho(r)-1} .$$

Taking limits we get

$$\sigma \leq \alpha. \tag{3.14}$$

Combining this with (3.11), (3.12) and (3.14) we obtain

$$\sigma \leq \beta$$

which gives a contradiction. Hence strict inequality does not exist. ∎

Remark 3.1: It is interesting to mention that if $(p,q) = (2,1)$ and $n_k = k$, $\forall\, k \in \mathbf{N}$, this theorem includes Theorem 1.23 of [7]. ∎

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References

[1] S. K. Bose and D. Sharma, *Integral functions of two complex variables*, Compositio Mathematica **15** (1963), 210–226.

[2] A. A. Goldberg, *Elementary remarks on the formulas defining order and type of functions of several variables* (Russian), Doklady Akademii Nauk Armyanskii SSR **29** (1959), 145–151.

[3] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the (p,q)-order and lower (p,q)-order of an entire function*, Journal für die Reine und Angewandte Mathematik **282** (1976), 53–67.

[4] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the (p,q)-type and lower (p,q)-type of an entire function*, Journal für die Reine und Angewandte Mathematik **290** (1977), 180–190.

[5] H. S. Kasana, *The existence theorem for proximate order of analytic functions*, Ukrainskii Matematichnii Zhurnal (Kiev) **40** (1988), no. 1, 117–121, 135; English translation as Ukrainian Mathematical Journal (New York), 100–103.

[6] H. S. Kasana and A. Sahai, *The proximate order of entire Dirichlet series*, Complex Variables: Theory and Application (New York) **9** (1987), no. 1, 49–62.

[7] P. Lelong and L. Gruman, *Entire Functions of Several Complex Variables*, Grundlehren der mathematischen Wissenschaften, **282**, Springer-Verlag, Berlin, 1986.

[8] B. Ja. Levi, *Distribution of Zeros of entire functions*, Translations of Mathematics Monographs **55**. AMS, Providence, RI (1964) (published in Russian in 1956).

[9] P. Lockhart and E. G. Straus, *Relations between maximum modulus and maximum term of an entire function*, Pacific Journal of Mathematics **118** (1985), 479–485.

[10] K. Nandan, R. P. Doherey and R. S. L. Srivastava, *Proximate order of an entire function with index-pair (p,q)*, Indian Journal of Pure and Applied Mathematics **11** (1963), 33–39.

[11] G. Valiron, *Lectures on the general theory of integral functions*, Private, Toulouse (1922).